Multiple $\mathbb{S}^1$-orbits for the Schrödinger-Newton system

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Abstract

We prove existence and multiplicity of symmetric solutions for the *Schrödinger-Newton system* in three dimensional space using equivariant Morse theory.

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1 Introduction

The *Schrödinger-Newton system* in three dimensional space has a long standing history. It was firstly proposed in 1954 by Pekar for describing the quantum mechanics of a polaron. Successively it was derived by Choquard for describing an electron trapped in its own hole and by Penrose [27, 28, 29] in his discussions on the selfgravitating matter.

For a single particle of mass m the system is obtained by coupling together the linear Schrödinger equation of quantum mechanics with the Poisson equation from Newtonian mechanics. It has the form

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi + U\psi = 0, \\ -\Delta U + 4\pi\kappa|\psi|^2 = 0, \end{cases} \tag{1}$$

where ψ is the complex wave function, U is the gravitational potential energy, V is a given potential, $\hbar$ is Planck's constant, and $\kappa := \mathrm{G}m^2$, G being Newton's constant.

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Rescaling $\psi(x)=\frac{1}{\hbar}\frac{\hat{\psi}(x)}{\sqrt{2\kappa m}}$, $V(x)=\frac{1}{2m}\hat{V}(x)$, $U(x)=\frac{1}{2m}\hat{U}(x)$, system (1) becomes equivalent to the single nonlocal equation

$$-\hbar^2\Delta\hat{\psi}+\hat{V}(x)\hat{\psi}=\frac{1}{\hbar^2}\Big(\frac{1}{|x|}*|\hat{\psi}|^2\Big)\hat{\psi}. \tag{2}$$

The existence of one solution can be traced back to Lions' paper [19]. Successively equation (2) and related equations have been investigated by many authors, see e.g. [2, 12, 16, 13, 15, 20, 21, 24, 22, 25, 30, 31, 8, 23] and the references therein. Semiclassical analysis for equation (2) has been studied in [33] and in [10] for a more general convolution potential, not necessarily radially symmetric.

In this work we shall consider the nonlocal equation (2) in presence of a magnetic potential A and an electric potential V which satisfy specific symmetry. Precisely, we consider G a closed subgroup of the group $O(3)$ of linear isometries of $\mathbb{R}^3$ and assume that $A\colon \mathbb{R}^3\to\mathbb{R}^3$ is a C^1-function, and $V\colon \mathbb{R}^3\to\mathbb{R}$ is a bounded continuous function with $\inf_{\mathbb{R}^3}V>0$, which satisfy

$$A(gx)=gA(x)\quad\text{and}\quad V(gx)=V(x)\qquad\text{for all } g\in G,\, x\in\mathbb{R}^3. \tag{3}$$

Given a continuous homomorphism of groups $\tau\colon G\to\mathbb{S}^1$ into the group $\mathbb{S}^1$ of unit complex numbers. A physically relevant example is a constant magnetic field $B=\operatorname{curl}A=(0,0,2)$ and the group $G_m=\{e^{2\pi ik/m}\mid k=1,\dots,m\}$ for $m\in\mathbb{N}$, $m\geq 1$; see Subsection 5.1 for more details.

We are interested in semiclassical states, i.e. solutions as $\varepsilon\to 0$ to the problem

$$\begin{cases}(-\varepsilon\mathrm{i}\nabla+A)^2u+V(x)u=\frac{1}{\varepsilon^2}\left(\frac{1}{|x|}*|u|^2\right)u,\\ u\in L^2(\mathbb{R}^3,\mathbb{C}),\\ \varepsilon\nabla u+\mathrm{i}Au\in L^2(\mathbb{R}^3,\mathbb{C}^3),\end{cases} \tag{4}$$

which satisfy

$$u(gx)=\tau(g)u(x)\qquad\text{for all } g\in G,\ x\in\mathbb{R}^3. \tag{5}$$

This implies that the absolute value $|u|$ of u is G-invariant and the phase of $u(gx)$ is that of $u(x)$ multiplied by $\tau(g)$.

Recently in [9] the authors have showed that there is a combined effect of the symmetries and the electric potential V on the number of semiclassical τ-intertwining solutions to (4). More precisely, we showed that the Lusternik-Schnirelmann category of the G-orbit space of a suitable set M_τ, depending on V and τ, furnishes a lower bound on the number of solutions of this type. In this work we shall apply equivariant Morse theory for better multiplicity results than those given by Lusternik-Schnirelmann category. Moreover equivariant Morse theory provides information on the local behavior of a functional around a critical orbit. The main result is established in Theorem 5.3. For the local case, similar results are obtained in [7]. For other results about magnetic Schrödinger equations, we refer to [4, 5].

Finally, concerning magnetic Pekar functional, we mention the recent results in [14].

2 The variational problem

Set $\nabla_{\varepsilon,A}u=\varepsilon\nabla u+\mathrm{i}Au$ and consider the real Hilbert space

$$H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C}):=\{u\in L^2(\mathbb{R}^3,\mathbb{C})\mid \nabla_{\varepsilon,A}u\in L^2(\mathbb{R}^3,\mathbb{C}^3)\}$$

with the scalar product

$$\langle u,v\rangle_{\varepsilon,A,V}=\operatorname{Re}\int_{\mathbb{R}^3}\left(\nabla_{\varepsilon,A}u\cdot\overline{\nabla_{\varepsilon,A}v}+V(x)u\bar{v}\right). \tag{6}$$

We write

$$\|u\|_{\varepsilon,A,V} = \Big(\int_{\mathbb{R}^3} \Big(|\nabla_{\varepsilon,A} u|^2 + V(x)\,|u|^2 \Big) \Big)^{1/2}$$

for the corresponding norm.

If $u \in H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C})$, then $|u| \in H^1(\mathbb{R}^3,\mathbb{R})$ and the diamagnetic inequality [18] holds

$$\varepsilon\, |\nabla |u(x)|| \leq |\varepsilon \nabla u(x) + \mathrm{i} A(x) u(x)| \quad \text{for a.e. } x \in \mathbb{R}^3. \tag{7}$$

Set

$$\mathbb{D}(u) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|}\, dxdy.$$

We need some basic inequalities about convolutions. A proof can be found in [18, Theorem 4.3] and in [17].

Theorem 2.1. *If $p,\, q \in (1,+\infty)$ satisfy $1/p + 1/3 = 1 + 1/q$ and $f \in L^p(\mathbb{R}^3)$ then*

$$\| |x| * f \|_{L^q(\mathbb{R}^3)} \leq N_p \|f\|_{L^p(\mathbb{R}^3)} \tag{8}$$

for a constant $N_p > 0$ that depends on p but not on f. More generally, if $p,\ t \in (1,+\infty)$ satisfy $1/p + 1/t + 1/3 = 2$ and $f \in L^p(\mathbb{R}^3)$, $g \in L^t(\mathbb{R}^3)$, then

$$\Big| \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{f(x) g(y)}{|x-y|}\, dx\, dy \Big| \leq N_p \|f\|_{L^p(\mathbb{R}^3)} \|g\|_{L^t(\mathbb{R}^3)} \tag{9}$$

for some constant $N_p > 0$ that does not depend on f and g.

Theorem 2.1 yields that

$$\mathbb{D}(u) \leq C \|u\|^4_{L^{12/5}(\mathbb{R}^3)} \tag{10}$$

for every $u \in H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C})$.

The energy functional $J_{\varepsilon,A,V} \colon H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C}) \to \mathbb{R}$ associated to problem (4), defined by

$$J_{\varepsilon,A,V}(u) = \frac{1}{2} \|u\|^2_{\varepsilon,A,V} - \frac{1}{4\varepsilon^2} \mathbb{D}(u),$$

is of class C^1, and its first derivative is given by

$$J'_{\varepsilon,A,V}(u)[v] = \langle u, v \rangle_{\varepsilon,A,V} - \frac{1}{\varepsilon^2} \operatorname{Re} \int_{\mathbb{R}^3} \Big(\frac{1}{|x|} * |u|^2 \Big) u \bar{v}.$$

Moreover we can write the second derivative

$$J''_{\varepsilon,A,V}(u)[v,w] = \langle w, v \rangle_{\varepsilon,A,V} - \frac{1}{\varepsilon^2} \operatorname{Re} \int_{\mathbb{R}^3} \Big(\frac{1}{|x|} * |u|^2 \Big) w \bar{v} - \frac{2}{\varepsilon^2} \operatorname{Re} \int_{\mathbb{R}^3} \Big(\frac{1}{|x|} * (u \bar{w}) \Big) u \bar{v}.$$

By (9) it is easy to recognize that

$$\begin{aligned} |J''_{\varepsilon,A,V}(u)[v,w]| &\leq \|v\|_{\varepsilon,A,V}\ \|w\|_{\varepsilon,A,V} + C \|u\|^2_{L^{12/5}(\mathbb{R}^3)} \|v\|_{L^{12/5}(\mathbb{R}^3)} \|w\|_{L^{12/5}(\mathbb{R}^3)} \\ &\leq K \|v\|_{\varepsilon,A,V}\ \|w\|_{\varepsilon,A,V}. \end{aligned}$$

We postpone the proof that $J_{\varepsilon,A,V}$ is of class C^2 to the Appendix.

The solutions to problem (4) are the critical points of $J_{\varepsilon,A,V}$. The action of G on $H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C})$ defined by $(g,u)\mapsto u_g$, where

$$(u_g)(x)=\tau(g)u(g^{-1}x),$$

satisfies

$$\langle u_g,v_g\rangle_{\varepsilon,A,V}=\langle u,v\rangle_{\varepsilon,A,V}\quad\text{and}\quad\mathbb{D}(u_g)=\mathbb{D}(u)$$

for all $g\in G$, $u,v\in H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C})$. Hence, $J_{\varepsilon,A,V}$ is G-invariant. By the principle of symmetric criticality [26, 34], the critical points of the restriction of $J_{\varepsilon,A,V}$ to the fixed point space of this G-action, denoted by

$$\begin{aligned}H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C})^\tau&=\left\{u\in H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C})\mid u_g=u\right\}\\&=\left\{u\in H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C})\mid u(gx)=\tau(g)u(x)\quad\forall x\in\mathbb{R}^3,\ g\in G\right\},\end{aligned}$$

are the solutions to problem (4) which satisfy (5).

Let us define the *Nehari manifold*

$$\mathcal{N}^\tau_{\varepsilon,A,V}=\left\{u\in H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C})^\tau\mid u\neq 0\text{ and }\varepsilon^2\,\|u\|^2_{\varepsilon,A,V}=\mathbb{D}(u)\right\},$$

which is a C^2-manifold radially diffeomorphic to the unit sphere in $H^1_{\varepsilon,A}(\mathbb{R}^3,\ \mathbb{C})^\tau$. The critical points of the restriction of $J_{\varepsilon,A,V}$ to $\mathcal{N}^\tau_{\varepsilon,A,V}$ are precisely the nontrivial solutions to (4) which satisfy (5).

Since $\mathbb{S}^1$ acts on $H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C})^\tau$ by scalar multiplication: $(e^{\mathrm{i}\theta},u)\mapsto e^{\mathrm{i}\theta}u$, the Nehari manifold $\mathcal{N}^\tau_{\varepsilon,A,V}$ and the functional $J_{\varepsilon,A,V}$ are invariant under this action. Therefore, if u is a critical point of $J_{\varepsilon,A,V}$ on $\mathcal{N}^\tau_{\varepsilon,A,V}$ then so is γu for every $\gamma\in\mathbb{S}^1$. The set $\mathbb{S}^1u=\{\gamma u\mid\gamma\in\mathbb{S}^1\}$ is then called a *τ-intertwining critical $\mathbb{S}^1$-orbit of* $J_{\varepsilon,A,V}$. Two solutions of (4) are said to be *geometrically different* if their $\mathbb{S}^1$-orbits are different.

Recall that $J_{\varepsilon,A,V}:\mathcal{N}^\tau_{\varepsilon,A,V}\to\mathbb{R}$ is said to satisfy the *Palais-Smale condition* $(PS)_c$ at the level c if every sequence (u_n) such that

$$u_n\in\mathcal{N}^\tau_{\varepsilon,A,V},\quad J_{\varepsilon,A,V}(u_n)\to c,\quad\nabla_{\mathcal{N}^\tau_{\varepsilon,A,V}}J_{\varepsilon,A,V}(u_n)\to 0$$

contains a convergent subsequence. Here $\nabla_{\mathcal{N}^\tau_{\varepsilon,A,V}}J_{\varepsilon,A,V}(u)$ denotes the orthogonal projection of $\nabla_\varepsilon J_{\varepsilon,A,V}(u)$ onto the tangent space to $\mathcal{N}^\tau_{\varepsilon,A,V}$ at u.

In Lemma 3.4 of [8] the following result was proved for $\varepsilon=1$.

Proposition 2.2. *For every $\varepsilon>0$, the functional $J_{\varepsilon,A,V}:\mathcal{N}^\tau_{\varepsilon,A,V}\to\mathbb{R}$ satisfies $(PS)_c$ at each level*

$$c<\varepsilon^3\min_{x\in\mathbb{R}^3\setminus\{0\}}(\#Gx)V_\infty^{3/2}E_1,$$

where $V_\infty=\liminf_{|x|\to\infty}V(x)$.

3 The limit problem

For any positive real number λ we consider the problem

$$\begin{cases}-\Delta u+\lambda u=(\frac{1}{|x|}*u^2)u,\\u\in H^1(\mathbb{R}^3,\mathbb{R}).\end{cases}\tag{11}$$

Its associated energy functional $J_\lambda : H^1(\mathbb{R}^3,\mathbb{R}) \to \mathbb{R}$ is given by

$$J_\lambda(u) = \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{4}\mathbb{D}(u), \quad \text{with} \ \ \|u\|_\lambda^2 = \int_{\mathbb{R}^3}\left(|\nabla u|^2 + \lambda u^2\right).$$

Its Nehari manifold will be denoted by

$$\mathscr{M}_\lambda = \left\{u \in H^1(\mathbb{R}^3,\mathbb{R}) \mid u \neq 0, \quad \|u\|_\lambda^2 = \mathbb{D}(u)\right\}.$$

We set

$$E_\lambda = \inf_{u\in\mathscr{M}_\lambda} J_\lambda(u).$$

The critical points of J_λ on $\mathscr{M}_\lambda$ are the nontrivial solutions to (11). Note that u solves the real-valued problem

$$\begin{cases} -\Delta u + u = (\frac{1}{|x|} * u^2)u, \\ u \in H^1(\mathbb{R}^3,\mathbb{R}) \end{cases} \tag{12}$$

if and only if $u_\lambda(x) = \lambda u(\sqrt{\lambda}x)$ solves (11). Therefore,

$$E_\lambda = \lambda^{3/2}E_1.$$

where E_1 is the least energy of a nontrivial solution to (12). Minimizers of J_λ on $\mathscr{M}_\lambda$ are called ground states. The existence and uniqueness of ground states up to sign and translations was established by Lieb in [16]. We denote by ω_λ the positive solution to problem (11) which is radially symmetric with respect to the origin.

Fix a radial function $\rho \in C^\infty(\mathbb{R}^3,\mathbb{R})$ such that $\rho(x) = 1$ if $|x| \leq \frac{1}{2}$ and $\rho(x) = 0$ if $|x| \geq 1$. For $\varepsilon > 0$ set $\rho_\varepsilon(x) = \rho(\sqrt{\varepsilon}x)$, $\omega_{\lambda,\varepsilon} = \rho_\varepsilon\omega_\lambda$ and

$$\upsilon_{\lambda,\varepsilon} = \frac{\|\omega_{\lambda,\varepsilon}\|_\lambda}{\sqrt{\mathbb{D}(\omega_{\lambda,\varepsilon})}}\,\omega_{\lambda,\varepsilon}. \tag{13}$$

Note that $\operatorname{supp}(\upsilon_{\lambda,\varepsilon}) \subset B(0,1/\sqrt{\varepsilon}) = \{x \in \mathbb{R}^3 \mid |x| \leq 1/\sqrt{\varepsilon}\}$ and $\upsilon_{\lambda,\varepsilon} \in \mathscr{M}_\lambda$. An easy computation shows that

$$\lim_{\varepsilon\to 0} J_\lambda(\upsilon_{\lambda,\varepsilon}) = \lambda^{3/2}E_1. \tag{14}$$

Now we define

$$\ell_{G,V} = \inf_{x\in\mathbb{R}^N}(\#Gx)V^{3/2}(x)$$

and consider the set

$$M_\tau = \{x \in \mathbb{R}^N \mid (\#Gx)V^{3/2}(x) = \ell_{G,V},\ G_x \subset \ker\tau\}.$$

Here $Gx = \{gx \mid g \in G\}$ is the G-orbit of the point $x \in \mathbb{R}^3$, $\#Gx$ is its cardinality, and $G_x = \{g \in G \mid gx = x\}$ is its isotropy subgroup. Observe that the points in M_τ are not necessarily local minima of V.

In what follows we will assume that there exists $\alpha > 0$ such that the set

$$\left\{y \in \mathbb{R}^3 \mid (\#Gy)V^{3/2}(y) \leq \ell_{G,V} + \alpha\right\}$$

is compact. Then

$$M_{G,V} = \left\{y \in \mathbb{R}^3 \mid (\#Gy)V^{3/2}(y) = \ell_{G,V}\right\}$$

is a compact G-invariant set and all G-orbits in $M_{G,V}$ are finite. We split $M_{G,V}$ according to the orbit type of its elements, choosing subgroups $G_1, \dots, G_m$ of G such that the isotropy subgroup G_x of every point $x \in M_{G,V}$ is conjugate to precisely one of the G_i's, and we set

$$M_i = \left\{ y \in M_{G,V} \mid G_y = gG_ig^{-1} \text{ for some } g \in G \right\}.$$

Since isotropy subgroups satisfy $G_{gx} = gG_xg^{-1}$, the sets M_i are G-invariant and, since V is continuous, they are closed and pairwise disjoint, and

$$M_{G,V} = M_1 \cup \cdots \cup M_m.$$

Moreover, since

$$|G/G_i|V^{3/2}(y) = (\#Gy)V^{3/2}(y) = \ell_{G,V} \quad \text{for all } \ y \in M_i,$$

the potential V is constant on each M_i. Here $|G/G_i|$ denotes the index of G_i in G. We denote by V_i the value of V on M_i.

It is well known that the map $G/G_\xi \to G\xi$ given by $gG_\xi \mapsto g\xi$ is a homeomorphism, see e.g. [11]. So, if $G_i \subset \ker \tau$ and $\xi \in M_i$, then the map

$$G\xi \to \mathbb{S}^1, \quad g\xi \mapsto \tau(g),$$

is well defined and continuous.

Let $\upsilon_{i,\varepsilon} = \upsilon_{V_i,\varepsilon}$ be defined as in (13) with $\lambda = V_i$. Set

$$\psi_{\varepsilon,\xi}(x) = \sum_{g\xi \in G\xi} \tau(g)\upsilon_{i,\varepsilon}\left(\frac{x - g\xi}{\varepsilon}\right)e^{-\mathrm{i}A(g\xi)\cdot\left(\frac{x-g\xi}{\varepsilon}\right)}. \tag{15}$$

Let $\pi_{\varepsilon,A,V} : H^1_{\varepsilon,A}(\mathbb{R}^3, \mathbb{C})^\tau \setminus \{0\} \to \mathcal{N}^\tau_{\varepsilon,A,V}$ be the radial projection given by

$$\pi_{\varepsilon,A,V}(u) = \frac{\varepsilon \|u\|_{\varepsilon,A,V}}{\sqrt{\mathbb{D}(u)}}u. \tag{16}$$

We can derive the following results, arguing as in Lemmas 2 in [6] (see also Lemma 4.2 in [9]).

Lemma 3.1. *Assume that $G_i \subset \ker \tau$. Then, the following hold:*

(a) For every $\xi \in M_i$ and $\varepsilon > 0$, one has that

$$\psi_{\varepsilon,\xi}(gx) = \tau(g)\psi_{\varepsilon,\xi}(x) \quad \forall g \in G, \ x \in \mathbb{R}^3.$$

(b) For every $\xi \in M_i$ and $\varepsilon > 0$, one has that

$$\tau(g)\psi_{\varepsilon,g\xi}(x) = \psi_{\varepsilon,\xi}(x) \quad \forall g \in G, \ x \in \mathbb{R}^3.$$

(c) One has that

$$\lim_{\varepsilon \to 0} \varepsilon^{-3} J_{\varepsilon,A,V}\left[\pi_{\varepsilon,A,V}(\psi_{\varepsilon,\xi})\right] = \ell_{G,V}E_1.$$

uniformly in $\xi \in M_i$.

Let

$$M_\tau = \{y \in M_{G,V} \mid G_y \subset \ker \tau\} = \bigcup_{G_i \subset \ker \tau} M_i.$$

As immediate consequence of Lemma 3.1, we derive the following result.

Proposition 3.2. *The map* $\widehat{\iota}_\varepsilon \colon M_\tau \to \mathscr{N}^\tau_{\varepsilon,A,V}$ *given by*

$$\widehat{\iota}_\varepsilon(\xi) = \pi_{\varepsilon,A,V}(\psi_{\varepsilon,\xi})$$

is well defined and continuous, and satisfies

$$\tau(g)\widehat{\iota}_\varepsilon(g\xi) = \widehat{\iota}_\varepsilon(\xi) \quad \forall \xi \in M_\tau,\ g \in G.$$

Moreover, given $d > \ell_{G,V} E_1$, *there exists* $\varepsilon_d > 0$ *such that*

$$\varepsilon^{-3} J_{\varepsilon,A,V}(\widehat{\iota}_\varepsilon(\xi)) \le d \quad \forall \xi \in M_\tau,\ \varepsilon \in (0,\varepsilon_d).$$

4 The baryorbit map

Let us consider the real-valued problem

$$\begin{cases} -\varepsilon^2 \Delta v + V(x) v = \frac{1}{\varepsilon^2}\left(\frac{1}{|x|} * u^2\right) u, \\ v \in H^1(\mathbb{R}^3,\mathbb{R}), \\ v(gx) = v(x) \ \ \forall x \in \mathbb{R}^3,\ g \in G. \end{cases} \tag{17}$$

Set

$$H^1(\mathbb{R}^3,\mathbb{R})^G = \{v \in H^1(\mathbb{R}^3,\mathbb{R}) \mid v(gx) = v(x)\ \forall x \in \mathbb{R}^3,\ g \in G\}$$

and write

$$\|v\|_V^2 = \int_{\mathbb{R}^3} \left(\varepsilon^2 |\nabla v|^2 + V(x) v^2\right).$$

The nontrivial solutions of (17) are the critical points of the energy functional

$$J_{\varepsilon,V}(v) = \frac{1}{2}\|v\|^2_{\varepsilon,V} - \frac{1}{4\varepsilon^2}\mathbb{D}(v)$$

on the Nehari manifold

$$\mathscr{M}^G_{\varepsilon,V} = \left\{ v \in H^1(\mathbb{R}^3,\mathbb{R})^G \mid v \neq 0,\ \|v\|^2_{\varepsilon,V} = \varepsilon^{-2}\mathbb{D}(v) \right\}.$$

Set

$$c^G_{\varepsilon,V} = \inf_{\mathscr{M}^G_{\varepsilon,V}} J_{\varepsilon,V} = \inf_{\substack{v \in H^1(\mathbb{R}^3,\mathbb{R})^G \\ v \neq 0}} \frac{\varepsilon^2 \|v\|^4_{\varepsilon,V}}{4\mathbb{D}(v)}. \tag{18}$$

As proved in Lemma 5.1 in [9] we have

Lemma 4.1. *There results* $0 < (\inf_{\mathbb{R}^3} V)^{3/2} E_1 \le \varepsilon^{-3} c^G_{\varepsilon,V}$ *for every* $\varepsilon > 0$, *and*

$$\limsup_{\varepsilon \to 0} \varepsilon^{-3} c^G_{\varepsilon,V} \le \ell_{G,V} E_1,$$

We fix $\hat{\rho} > 0$ such that

$$\begin{cases} |y - gy| > 2\hat{\rho} & \text{if } gy \neq y \in M_{G,W}, \\ \operatorname{dist}(M_i, M_j) > 2\hat{\rho} & \text{if } i \neq j, \end{cases} \tag{19}$$

where G_i, M_i, V_i are the groups, the sets and the values defined as in Section 3.

For $\rho \in (0, \hat{\rho})$, let

$$M_i^{\rho} = \{y \in \mathbb{R}^3 \colon \operatorname{dist}(y, M_i) \leq \rho, \;\; G_y = gG_ig^{-1} \text{ for some } g \in G\},$$

and for each $\xi \in M_i^{\rho}$ and $\varepsilon > 0$, define

$$\theta_{\varepsilon,\xi}(x) = \sum_{g\xi \in G\xi} \omega_i\Big(\frac{x - g\xi}{\varepsilon}\Big),$$

where ω_i is unique positive ground state of problem (11) with $\lambda = V_i$ which is radially symmetric with respect to the origin. Set

$$\Theta_{\rho,\varepsilon} = \left\{\theta_{\varepsilon,\xi} \mid \xi \in M_1^{\rho} \cup \cdots \cup M_m^{\rho}\right\}.$$

Arguing as in Proposition 5 in [8], we can derive the following result.

Proposition 4.2. *Given $\rho \in (0, \hat{\rho})$ there exist $d_\rho > \ell_{G,V} E_1$ and $\varepsilon_\rho > 0$ with the following property: For every $\varepsilon \in (0, \varepsilon_\rho)$ and every $v \in \mathscr{M}_{\varepsilon,V}^G$ with $J_{\varepsilon,V}(v) \leq \varepsilon^3 d_\rho$ there exists precisely one G-orbit $G\xi_{\varepsilon,v}$ with $\xi_{\varepsilon,v} \in M_1^{\rho} \cup \cdots \cup M_m^{\rho}$ such that*

$$\varepsilon^{-3} \left\| |v| - \theta_{\varepsilon,\xi_{\varepsilon,v}} \right\|_{\varepsilon,V}^2 = \min_{\theta \in \Theta_{\rho,\varepsilon}} \| |v| - \theta \|_{\varepsilon,V}^2.$$

For every $c \in \mathbb{R}$ we set

$$J_{\varepsilon,V}^c = \left\{v \in \mathscr{M}_{\varepsilon}^G \mid J_{\varepsilon,V}(v) \leq c\right\}.$$

Proposition 4.2 allows us to define, for each $\rho \in (0, \widehat{\rho})$ and $\varepsilon \in (0, \varepsilon_\rho)$, a local baryorbit map

$$\widehat{\beta}_{\rho,\varepsilon,0} \colon J_{\varepsilon,V}^{\varepsilon^3 d_\rho} \longrightarrow \left(M_1^{\rho} \cup \cdots \cup M_m^{\rho}\right)/G$$

by taking

$$\widehat{\beta}_{\rho,\varepsilon,0}(v) = G\xi_{\varepsilon,v},$$

where $G\xi_{\varepsilon,v}$ is the unique G-orbit given by the previous proposition.

Coming back to our original problem, for every $c \in \mathbb{R}$ set

$$J_{\varepsilon,A,V}^c = \{u \in \mathscr{N}_{\varepsilon,A,V}^{\tau} \mid J_{\varepsilon,A,V}(u) \leq c\}.$$

The following holds.

Corollary 4.3. *For each $\rho \in (0, \widehat{\rho})$ and $\varepsilon \in (0, \varepsilon_\rho)$, the local baryorbit map*

$$\widehat{\beta}_{\rho,\varepsilon} \colon J_{\varepsilon,A,V}^{\varepsilon^3 d_\rho} \longrightarrow \left(M_1^{\rho} \cup \cdots \cup M_m^{\rho}\right)/G,$$

given by

$$\widehat{\beta}_{\rho,\varepsilon}(u) = \widehat{\beta}_{\rho,\varepsilon,0}(\hat{\pi}_\varepsilon(|u|)),$$

where $\hat{\pi}_\varepsilon : H^1(\mathbb{R}^3,\mathbb{R})^G\setminus\{0\}\to\mathscr{M}_\varepsilon^G$ *is the radial projection, is well defined and continuous. It satisfies*

$$\widehat{\beta}_{\rho,\varepsilon}(\gamma u)=\widehat{\beta}_{\rho,\varepsilon}(u)\quad \forall\gamma\in\mathbb{S}^1,$$
$$\widehat{\beta}_{\rho,\varepsilon}(\widehat{\iota}_\varepsilon(\xi))=\xi\quad \forall\xi\in M_\tau \text{ with } J_{\varepsilon,A,V}(\iota_\varepsilon(\xi))\leq\varepsilon^3 d_\rho,$$

where $\widehat{\iota}_\varepsilon$ *is the map defined in* Proposition 3.2.

Proof. If $u\in\mathscr{N}^\tau_{\varepsilon,A,V}$ then $\hat{\pi}_\varepsilon(|u|)\in\mathscr{M}_\varepsilon^G$. The diamagnetic inequality yields

$$J_{\varepsilon,V}(\hat{\pi}_\varepsilon(|u|))\leq J_{\varepsilon,A,V}(u). \tag{20}$$

So if $J_{\varepsilon,A,V}(u)\leq\varepsilon^3 d_\rho$ then $\widehat{\beta}_{\rho,\varepsilon}(u)$ is well defined. It is straightforward to verify that it has the desired properties. □

Corollary 4.4. *If there exists* $\xi\in\mathbb{R}^3$ *such that* $(\#G\xi)V^{3/2}(\xi)=\ell_{G,V}$ *and* $G_\xi\subset\ker\tau$*, then*

$$\lim_{\varepsilon\to\infty}\varepsilon^{-3}c^\tau_{\varepsilon,A,V}=\ell_{G,V}E_1,$$

where $c^\tau_{\varepsilon,A,V}=\inf_{\mathscr{N}^\tau_{\varepsilon,A,V}}J_{\varepsilon,A,V}$.

Proof. Inequality (20) yields $c^G_{\varepsilon,V}=\inf_{\mathscr{M}^G_{\varepsilon,V}}J_{\varepsilon,V}\leq\inf_{\mathscr{N}^\tau_{\varepsilon,A,V}}J_{\varepsilon,A,V}=c^\tau_{\varepsilon,A,V}$. By statement (c) of Lemma 3.1,

$$\ell_{G,V}E_1=\lim_{\varepsilon\to\infty}\varepsilon^{-3}c^G_{\varepsilon,V}\leq\liminf_{\varepsilon\to 0}\varepsilon^{-3}c^\tau_{\varepsilon,A,V}\leq\limsup_{\varepsilon\to\infty}\varepsilon^{-3}c^\tau_{\varepsilon,A,V}\leq\ell_{G,V}E_1.$$ □

5 Multiplicity results via Equivariant Morse theory

We start by reviewing some well known facts on equivariant Morse theory. We refer the reader to [3, 32] for further details.

Definition 5.1. Let Γ be a compact Lie group and X be a Γ-space.

- The Γ-orbit of a point $x\in X$ is the set $\Gamma x:=\{\gamma x\mid\gamma\in\Gamma\}$.
- A subset A of X is said to be Γ-invariant if $\Gamma x\subset A$ for every $x\in A$. The Γ-orbit space of A is the set $A/\Gamma:=\{\Gamma x: x\in A\}$ with the quotient space topology.
- X is called a free Γ-space if $\gamma x\neq x$ for every $\gamma\in\Gamma$, $x\in X$.
- A map $f\colon X\to Y$ between Γ-spaces is called Γ-invariant if f is constant on each Γ-orbit of X, and it is called Γ-equivariant if $f(\gamma x)=\gamma f(x)$ for every $\gamma\in\Gamma$, $x\in X$.

We fix a field $\mathbb{K}$ and denote by $\mathscr{H}^*(X,A)$ the Alexander-Spanier cohomology of the pair (X,A) with coefficients in $\mathbb{K}$. If X is a Γ-pair, i.e. if X is a Γ-space and A is a Γ-invariant subset of X, we write

$$\mathscr{H}^*_\Gamma(X,A):=\mathscr{H}^*(E\Gamma\times_\Gamma X,E\Gamma\times_\Gamma A)$$

for the Borel-cohomology that pair. $E\Gamma$ is the total space of the classifying Γ-bundle and $E\Gamma\times_\Gamma X$ is the orbit space $(E\Gamma\times X)/\Gamma$ (see e.g. [11, Chapter III]). If X is a free Γ-space, as will be the case in our application, then the projection $E\Gamma\times_\Gamma X\to X/\Gamma$ is a homotopy equivalence and it induces an isomorphism

$$\mathscr{H}^*_\Gamma(X,A)\cong\mathscr{H}^*(X/\Gamma,A/\Gamma). \tag{21}$$

In our setting, $\Gamma = \mathbb{S}^1$; if $A \subset X$ are $\mathbb{S}^1$-invariant subsets of $\mathscr{N}^\tau_{\varepsilon,A,V}$ we denote by $X/\mathbb{S}^1$ and $A/\mathbb{S}^1$ their $\mathbb{S}^1$-orbit spaces and by (21) it is legitimate to write

$$\mathscr{H}^*_{\mathbb{S}^1}(X,A) \simeq \mathscr{H}^*(X/\mathbb{S}^1, A/\mathbb{S}^1).$$

If $\mathbb{S}^1 u$ is an isolated critical $\mathbb{S}^1$-orbit of $J_{\varepsilon,A,V}$ its *k-th critical group* is defined as

$$C^k_{\mathbb{S}^1}(J_{\varepsilon,A,V}, \mathbb{S}^1 u) = \mathscr{H}^k_{\mathbb{S}^1}(J^c_{\varepsilon,A,V} \cap U, (J^c_{\varepsilon,A,V} \smallsetminus \mathbb{S}^1 u) \cap U),$$

where U is an $\mathbb{S}^1$-invariant neighborhood of $\mathbb{S}^1 u$ in $\mathscr{N}^\tau_{\varepsilon,A,V}$, $c = J_{\varepsilon,A,V}(u)$. Its total dimension

$$\mu(J_\varepsilon, \mathbb{S}^1 u) = \sum_{k=0}^{\infty} \dim C^k_{\mathbb{S}^1}(J_{\varepsilon,A,V}, \mathbb{S}^1 u)$$

is called the *multiplicity of* $\mathbb{S}^1 u$. If $\mathbb{S}^1 u$ is nondegenerate and $J_{\varepsilon,A,V}$ satisfies the Palais-Smale condition in some neighborhood of c, then

$$\dim C^k_{\mathbb{S}^1}(J_{\varepsilon,A,V}, \mathbb{S}^1 u) = 1$$

if k is the Morse index of $J_{\varepsilon,A,V}$ at the critical submanifold $\mathbb{S}^1 u$ of $\mathscr{N}^\tau_{\varepsilon,A,V}$ and it is 0 otherwise.

Moreover, for $\rho > 0$ we set

$$B_\rho M_\tau = \{x \in \mathbb{R}^3 \mid \operatorname{dist}(x, M_\tau) \leq \rho\}$$

and write $i_\rho \colon M_\tau/G \hookrightarrow B_\rho M_\tau/G$ for the embedding of the G-orbit space of M_τ in that of $B_\rho M_\tau$. We will show that this embedding has an effect on the number of solutions of (4) for ε small enough.

Lemma 5.2. *For every $\rho \in (0, \widehat{\rho})$ and $d \in (\ell_{G,V} E_1, d_\rho)$, with d_ρ as in* Proposition 4.2*, there exists $\varepsilon_{\rho,d} > 0$ such that*

$$\dim \mathscr{H}^k(J^{\varepsilon^3 d}_{\varepsilon,A,V}/\mathbb{S}^1) \geq \operatorname{rank}\left(i^*_\rho \colon \mathscr{H}^k(B_\rho M_\tau/G) \to \mathscr{H}^k(M_\tau/G)\right)$$

for every $\varepsilon \in (0, \varepsilon_{\rho,d})$ and $k \geq 0$, where $i_\rho \colon M_\tau/G \hookrightarrow B_\rho M_\tau/G$ is the inclusion map.

Proof. Let $\varepsilon_{\rho,d} = \min\{\varepsilon_d, \varepsilon_\rho\}$ where ε_ρ is as in Proposition 4.2 and ε_d is as in Proposition 3.2. Fix $\varepsilon \in (0, \varepsilon_{\rho,d})$. Then,

$$J_{\varepsilon,A,V}(\widehat{\iota}_\varepsilon(\xi)) \leq \varepsilon^3 d \quad \text{and} \quad \widehat{\beta}_{\rho,\varepsilon}(\widehat{\iota}_\varepsilon(\xi)) = \xi \qquad \forall \xi \in M_\tau.$$

By Proposition 3.2 and Corollary 4.3 the maps

$$M_\tau/G \xrightarrow{\iota_\varepsilon} J^{\varepsilon^3 d}_{\varepsilon,A,V}/\mathbb{S}^1 \xrightarrow{\beta_{\rho,\varepsilon}} B_\rho M/G$$

given by $\iota_\varepsilon(G\xi) = \widehat{\iota}_\varepsilon(\xi)$ and $\beta_{\rho,\varepsilon}(\mathbb{S}^1 u) = \widehat{\beta}_{\rho,\varepsilon}(u)$ are well defined and satisfy $\beta_{\rho,\varepsilon}(\iota_\varepsilon(G\xi)) = G\xi$ for all $\xi \in M_\tau$. Note that $M_\tau = \bigcup\{M_i \mid G_i \subset \ker \tau\}$ is the union of some connected components of M. Moreover, our choice of $\widehat{\rho}$ implies that $B_\rho M_\tau \cap B_\rho (M \smallsetminus M_\tau) = \emptyset$. Therefore the inclusion $i_{\tau,\rho} \colon B_\rho M_\tau/G \hookrightarrow B_\rho M/G$ induces an epimorphism in cohomology. Since $\beta_{\rho,\varepsilon} \circ \iota_\varepsilon = i_{\tau,\rho} \circ i_\rho$ we conclude that

$$\begin{aligned}
\dim \mathscr{H}^k(J^{\varepsilon^3 d}_{\varepsilon,A,V}/\mathbb{S}^1) &\geq \operatorname{rank}(\iota^*_\varepsilon \colon \mathscr{H}^k(J^{\varepsilon^3 d}_\varepsilon/\mathbb{S}^1) \to \mathscr{H}_k(M_\tau/G)) \\
&\geq \operatorname{rank}((\beta_{\rho,\varepsilon} \circ \iota_\varepsilon)^* \colon \mathscr{H}^k(B_\rho M/G) \to \mathscr{H}_k(M_\tau/G)) \\
&= \operatorname{rank}\left(i^*_\rho \colon \mathscr{H}^k(B_\rho M_\tau/G) \to \mathscr{H}^k(M_\tau/G)\right),
\end{aligned}$$

as claimed. $\square$

We are ready to prove our main theorem.

Theorem 5.3. *Assume there exists $\alpha > 0$ such that the set*

$$\{x \in \mathbb{R}^3 \mid (\#Gx)V^{3/2}(x) \leq \ell_{G,V} + \alpha\}. \tag{22}$$

is compact. Then, given $\rho > 0$ and $\delta \in (0, \alpha)$, there exists $\overline{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \overline{\varepsilon})$ one of the following two assertions holds:

- *(a) $J_{\varepsilon,A,V}$ has a nonisolated τ-intertwining critical $\mathbb{S}^1$-orbit in the set $J^{-1}_{\varepsilon,A,V}[\varepsilon^3(\ell_{G,V}E_1 - \delta), \varepsilon^3(\ell_{G,V}E_1 + \delta)]$.*
- *(b) $J_{\varepsilon,A,V}$ has finitely many τ-intertwining critical $\mathbb{S}^1$-orbits $\mathbb{S}^1u_1, \mathbb{S}^1u_2, \dots, \mathbb{S}^1u_m$ in $J^{-1}_{\varepsilon,A,V}[\varepsilon^3(\ell_{G,V}E_1 - \delta), \varepsilon^3(\ell_{G,V}E_1 + \delta)]$. They satisfy*

$$\sum_{j=1}^{m} \dim C^k_{\mathbb{S}^1}(J_{\varepsilon,A,V}, \mathbb{S}^1u_j) \geq \operatorname{rank}(i^*_\rho : \mathscr{H}^k(B_\rho M_\tau/G) \to \mathscr{H}^k(M_\tau/G))$$

  for every $k \geq 0$.

In particular, if every τ-intertwining critical $\mathbb{S}^1$-orbit of $J_{\varepsilon,A,V}$ in the set $J^{-1}_{\varepsilon,A,V}[\varepsilon^3(\ell_{G,V}E_1 - \delta), \varepsilon^3(\ell_{G,V}E_1 + \delta)]$ is nondegenerate then, for every $k \geq 0$, there are at least

$$\operatorname{rank}(i^*_\rho : \mathscr{H}^k(B_\rho M_\tau/G) \to \mathscr{H}^k(M_\tau/G))$$

of them having Morse index k for every $k \geq 0$.

Proof. Assume $M_\tau \neq \emptyset$ and let $\rho > 0$ and $\delta \in (0, \alpha E_1)$ be given. Without loss of generality we may assume that $\rho \in (0, \overline{\rho})$. Assumption (22) implies that

$$\ell_{G,V} + \alpha \leq \min_{x \in \mathbb{R}^3 \smallsetminus \{0\}} (\#Gx)V_\infty^{3/2}$$

where $V_\infty = \limsup_{|x| \to \infty} V(x)$. By Proposition 2.2 the functional

$$J_{\varepsilon,A,V} : \mathscr{N}^\tau_{\varepsilon,A,V} \to \mathbb{R}$$

satisfies $(\mathrm{PS})_c$ at each level $c \leq \varepsilon^3(\ell_{G,V}E_1 + \delta)$ for every $\varepsilon > 0$. By Corollary 4.4 there exists $\varepsilon_0 > 0$ such that

$$\ell_{G,V}E_1 - \delta < \varepsilon^{-3} \inf_{u \in \mathscr{N}^\tau_\varepsilon} J_{\varepsilon,A,V} \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Let $d \in (\ell_{G,V}E_1, \min\{d_\rho, \ell_{G,V}E_1 + \delta\})$ with d_ρ as in Proposition 4.2, and $\overline{\varepsilon} = \min\{\varepsilon_0, \varepsilon_{\rho,d}\}$ with $\varepsilon_{\rho,d}$ as in Lemma 5.2. Fix $\varepsilon \in (0, \overline{\varepsilon})$ and for $u \in \mathscr{N}^\tau_{\varepsilon,A,V}$ with $J_{\varepsilon,A,V}(u) = c$ set

$$C^k_{\mathbb{S}^1}(J_{\varepsilon,A,V}, \mathbb{S}^1u) = \mathscr{H}^k((J^c_{\varepsilon,A,V} \cap U)/\mathbb{S}^1, ((J^c_{\varepsilon,A,V} \smallsetminus \mathbb{S}^1u) \cap U)/\mathbb{S}^1).$$

Assume that every critical $\mathbb{S}^1$-orbit of $J_{\varepsilon,A,V}$ lying in $J^{-1}_{\varepsilon,A,V}[\varepsilon^3(\ell_{G,V}E_1 - \delta), \varepsilon^3(\ell_{G,V}E_1 + \delta)]$ is isolated. Since $J_{\varepsilon,A,V} : \mathscr{N}^\tau_{\varepsilon,A,V} \to \mathbb{R}$ satisfies $(\mathrm{PS})_c$ at each $c \leq \varepsilon^3(\ell_{G,V}E_1 + \delta)$ there are only finitely many of them. Let $\mathbb{S}^1u_1, \dots, \mathbb{S}^1u_m$ be those critical $\mathbb{S}^1$-orbits of $J_{\varepsilon,A,V}$ in $\mathscr{N}^\tau_{\varepsilon,A,V}$ which satisfy $J_{\varepsilon,A,V}(u_i) < \varepsilon^3 d$.

Applying Theorem 7.6 in [3] to $J_{\varepsilon,A,V}: \mathscr{N}^{\tau}_{\varepsilon,A,V} \to \mathbb{R}$ with $a = \varepsilon^3(\ell_{G,V}E_1 - \delta)$ and $b = \varepsilon^3 d$ and Lemma 5.2 we obtain that

$$\sum_{j=1}^{m} \dim C^k_{\mathbb{S}^1}(J_{\varepsilon,A,V}, \mathbb{S}^1 u_i) \geq \dim \mathscr{H}^k(J^{\varepsilon^3 d}_{\varepsilon,A,V}/\mathbb{S}^1)$$
$$\geq \operatorname{rank}\left(i^*_\rho : \mathscr{H}^k(B_\rho M_\tau/G) \to \mathscr{H}^k(M_\tau/G)\right)$$

for every $k \geq 0$, as claimed. The last assertion of Theorem 5.3 is an immediate consequence of Theorem 7.6 in [3]. □

If the inclusion $i_\rho : M_\tau/G \hookrightarrow B_\rho M_\tau/G$ is a homotopy equivalence then

$$\operatorname{rank}\left(i^*_\rho : \mathscr{H}^k(B_\rho M_\tau/G) \to \mathscr{H}^k(M_\tau/G)\right) = \dim \mathscr{H}^k(M_\tau/G).$$

An immediate consequence of Theorem 5.3 is the following.

Corollary 5.4. *If assumption* (22) *holds then, given $\rho > 0$ and $\delta > 0$, there exists $\overline{\varepsilon} > 0$ such that for every $\varepsilon > 0$ problem* (4) *has at least*

$$\sum_{k=0}^{\infty} \operatorname{rank}(i^*_\rho : \mathscr{H}^k(B_\rho M_\tau/G) \to \mathscr{H}^k(M_\tau/G))$$

geometrically different solutions in $J^{-1}_{\varepsilon,A,V}[\varepsilon^3(\ell_{G,V}E_1 - \delta), \varepsilon^3(\ell_{G,V}E_1 + \delta)]$, counted with their multiplicity.

5.1 Examples

As a typical application of our existence result, we consider the constant magnetic field $B(x_1,x_2,x_3) = (0,0,2)$ in $\mathbb{R}^3$. We can consider its vector potential $A(x_1,x_2,x_3) = (-x_2,x_1,0)$, and identify $\mathbb{R}^3$ with $\mathbb{C}\times\mathbb{R}$. With this in mind, we write $A(z,t) = (iz,0)$, with $z = x_1 + ix_2$. We remark that $A(e^{i\theta}z,t) = e^{i\theta}A(z,t)$ for every $\theta \in \mathbb{R}$.

Given $m \in \mathbb{N}$, $m \geq 1$ and $n \in \mathbb{Z}$, we look for solutions u to problem (4) which satisfy the symmetry property

$$u\left(e^{2\pi ik/m}z,t\right) = e^{2\pi ink/m}u(z,t)$$

for every $k = 1,\dots,m$ and $(z,t) \in \mathbb{C}\times\mathbb{R}$. We assume that V satisfies

(a) $V \in C^2(\mathbb{R}^3)$ is bounded and $\inf_{\mathbb{R}^3} V > 0$; moreover

$$\inf_{x\in\mathbb{R}^3} V^{3/2}(x) < \liminf_{|x|\to+\infty} V^{3/2}(x).$$

(b) There exists $m_0 \in \mathbb{N}$ such that

$$m_0 \inf_{x\in\mathbb{R}^3} V^{3/2}(x) < \inf_{t\in\mathbb{R}} V^{3/2}(0,t)$$
$$V\left(e^{2\pi ik/m_0}z,t\right) = V(z,t)$$

for every $k = 1,\dots,m_0$ and $(z,t) \in \mathbb{C}\times\mathbb{R}$.

For each m that divides m_0 (in symbols: $m|m_0$), we consider the group

$$G_m = \left\{ e^{2\pi i k/m} \mid k = 1, \dots, m \right\}$$

acting by multiplication on the z-coordinate of each point $(z,t) \in \mathbb{C} \times \mathbb{R}$. It is easy to check that A and V match all the assumptions of Theorem 5.3 for each $G = G_m$: the compactness condition (22) follows from the two inequalities in (a) and (b). If $\tau\colon G_m \to \mathbb{S}^1$ is any homeomorphism, we have that

$$M_\tau = \left\{ x \in \mathbb{R}^3 \mid V(x) = \inf_{y \in \mathbb{R}^3} V(y) \right\}.$$

Given $n \in \mathbb{Z}$, we consider the homeomorphism $\tau\left(e^{2\pi i k/m}\right) = e^{2\pi i n k/m}$. In particular, given ρ, $\delta > 0$, for ε small enough we have

$$\sum_{m|m_0} \sum_{k=0}^{\infty} m \operatorname{rank}\left(i_\rho^* \colon \mathscr{H}^k(B_\rho M/G_m) \to \mathscr{H}^k(M/G_m) \right)$$

geometrically distinct solutions, counted with multiplicity.

Remark 1. Our multiplicity result cannot be obtained, in general, via standard category arguments. For a concrete example, consider $M = \bigcup_{n \geq 1} S_n$, where

$$S_n = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \left(x_1 - \frac{1}{n} \right)^2 + x_2^2 + x_3^2 = \frac{1}{n^2} \right\}.$$

The category of M is then 2, whereas

$$\lim_{\rho \to 0} \operatorname{rank}\left(i_\rho^* \colon \mathscr{H}^2(B_\rho M) \to \mathscr{H}^2(M) \right) = +\infty.$$

For a short proof, we refer to [7, Example 1, pag. 1280]

6 Appendix

Proposition 6.1. *The second derivative $J''_{\varepsilon,A,V}$ is continuous.*

Proof. We first prove that $J''_{\varepsilon,A,V}$ is continuous at zero. Let $\{u_n\}_n$ be a sequence in $H^1_{\varepsilon,A}(\mathbb{R}^3, \mathbb{C})$ converging to zero. By Sobolev's embedding theorem, $u_n \to 0$ in $L^r(\mathbb{R}^3)$ for $r \in [2,6]$. From (9) it follows that

$$\left| \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \frac{|u_n(y)|^2}{|x-y|} dy \right) w(x) \overline{v(x)} \, dx \right| \\ \leq C \|u_n\|^2_{L^{12/5}(\mathbb{R}^3)} \|v\|_{L^{12/5}(\mathbb{R}^3)} \|w\|_{L^{12/5}(\mathbb{R}^3)} \leq o(1) \|v\|_{\varepsilon,A,V} \ \|w\|_{\varepsilon,A,V} \quad (23)$$

This implies that

$$\lim_{n \to +\infty} \left| \operatorname{Re} \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \frac{|u_n(y)|^2}{|x-y|} dy \right) w(x) \overline{v(x)} \, dx \right| = 0 \quad (24)$$

whenever $u_n \to 0$ strongly in $H^1_{\varepsilon,A,V}(\mathbb{R}^3, \mathbb{C})$.

Similarly, we use (9) to prove that

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*(u_n\overline{v})\right)u_n\overline{w}\,dx\right| &= \left|\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{u_n(x)\overline{w(x)}u_n(y)\overline{v(y)}}{|x-y|}dx\,dy\right| \\ &\leq C\,\|u_n\overline{w}\|_{L^{6/5}(\mathbb{R}^3)}\,\|u_n\overline{v}\|_{L^{6/5}(\mathbb{R}^3)} \\ &\leq C\,\|u_n\|^2_{L^{12/5}(\mathbb{R}^3)}\,\|v\|_{L^{12/5}(\mathbb{R}^3)}\,\|w\|_{L^{12/5}(\mathbb{R}^3)}\end{aligned}$$

which implies that

$$\lim_{n\to+\infty}\left|\mathrm{Re}\int_{\mathbb{R}^3}\left(\frac{1}{|x|}*(u_n\overline{v})\right)u_n\overline{w}\,dx\right| = 0 \tag{25}$$

whenever $u_n \to 0$ strongly in $H^1_{\varepsilon,A,V}(\mathbb{R}^3,\mathbb{C})$. It is now easy to conclude that $J''_{\varepsilon,A,V}(u_n) \to J''_{\varepsilon,A,V}(0)$.

If $u_n \to u$ in $H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C})$, we replace $|u_n|^2$ in (23) with $u_n^0 = |u_n|^2 - |u_n-u|^2 - |u|^2$ and find

$$\left|\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{|u_n^0(y)|}{|x-y|}dy\right)w(x)\overline{v(x)}\,dx\right| \leq C\|u_n^0\|_{L^{6/5}(\mathbb{R}^3)}\|w\|_{\varepsilon,A,V}\|v\|_{\varepsilon,A,V} \leq o(1)\|w\|_{\varepsilon,A,V}\|v\|_{\varepsilon,A,V}.$$

Analogously

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{|u_n(y)-u(y)|^2}{|x-y|}dy\right)w(x)\overline{v(x)}\,dx\right| \\ &\qquad\leq C\|u_n-u\|^2_{L^{12/5}(\mathbb{R}^3)}\|w\|_{\varepsilon,A,V}\|v\|_{\varepsilon,A,V} \leq o(1)\|w\|_{\varepsilon,A,V}\|v\|_{\varepsilon,A,V},\end{aligned}$$

we conclude that

$$\left|\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{|u_n(y)|^2-|u(y)|^2}{|x-y|}dy\right)w(x)\overline{v(x)}\,dx\right| \leq o(1)\|w\|_{\varepsilon,A,V}\|v\|_{\varepsilon,A,V}. \tag{26}$$

Switching to the second term of $J''_{\varepsilon,A,V}(u_n) - J''_{\varepsilon,A,V}(u)$, we notice that

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{u_n(x)\overline{w(x)}u_n(y)\overline{v(y)}}{|x-y|}dx\,dy - \int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{u(x)\overline{w(x)}u(y)\overline{v(y)}}{|x-y|}dx\,dy \\ &\qquad= \int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\left[(u_n(y)-u(y))\,u_n(x)+(u_n(x)-u(x))\,u(y)\right]\overline{v(y)w(x)}}{|x-y|}dx\,dy,\end{aligned}$$

so that

$$\begin{aligned}&\left|\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{u_n(x)\overline{w(x)}u_n(y)\overline{v(y)}}{|x-y|}dx\,dy - \int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{u(x)\overline{w(x)}u(y)\overline{v(y)}}{|x-y|}dx\,dy\right| \\ &\qquad\leq \left|\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{((u_n(y)-u(y))\,u_n(x))\,\overline{v(y)w(x)}}{|x-y|}dx\,dy\right| \\ &\qquad\quad+\left|\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{((u_n(x)-u(x))\,u(y))\,\overline{v(y)w(x)}}{|x-y|}dx\,dy\right| \\ &\qquad\leq o(1)\,\|v\|_{\varepsilon,A,V}\,\|w\|_{\varepsilon,A,V}\end{aligned} \tag{27}$$

because $u_n \to u$. Recalling that $|\mathrm{Re}\,z| \leq |z|$ for every $z \in \mathbb{C}$ and putting together (26) and (27), we conclude that $J''_{\varepsilon,A,V}(u_n) \to J''_{\varepsilon,A,V}(u)$. □

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